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Oscillatory double-diffusive instabilities in a vertical slot

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Recent linear stability analyses of double-diffusive convection in a laterally heated vertical slot containing water have shown that for very weak (or no) vertical salinity gradient the initial instabilities are steady, but as the salinity gradient is increased there is a transition to oscillatory instabilities. For higher Prandtl number fluids the initial instabilities in a slot with no stratification can be oscillatory or wave-like. We show that the oscillatory instabilities in water are linked to these higher Prandtl number oscillatory instabilities. The salinity gradient has a destabilizing effect on these oscillations, making them appear for Prandtl numbers where oscillatory instabilities are not possible in the absence of salinity gradients. We derive an asymptotic description for this mode of instability.

1. Introduction

'Double-diffusive convection' refers to convection in a fluid where there are two diffusing components which have an effect on the density. The archetypal case is heat and salt. Convection that is dominated by the presence of two components is very common in geophysical systems, for example in the oceans and in magma chambers. A broad view of the subject of double-diffusive convection is given by Brandt & Fernando (1996).

One classic problem in double-diffusive convection is the lateral heating of a salt-stratified fluid in a vertical slot. This was first studied by Thorpe, Hutt & Soulsby (1969) who, in addition to conducting experiments, derived a theoretical criterion for the onset of instabilities for strong salinity gradients. This analysis was extended by Hart (1971) who also carried out a numerical investigation into the stability for a range of salt stratifications. A further numerical investigation was conducted by Thangam, Zebib & Chen (1981). Errors in their results for a relatively small range of gradients were found by Young & Rosner (1998) and Kerr & Tang (1999, hereafter referred to as KT). This range included a portion where the predicted instabilities were oscillatory. For further details the reader is referred to these papers.

The region on the stability boundary for double-diffusive convection in a vertical slot where the initial instability is oscillatory is indicated by the dashed line in figure 1 (from KT). (For the definitions of the heat and salt Rayleigh numbers, Ra_T and Ra_S, used in this figure, and of the Prandtl number and salt/heat diffusivity ratio see § 2.) This oscillatory instability occurs when the salinity stratification is relatively weak, but not too weak. These calculations were performed using a Prandtl number appropriate for water, $\sigma = 6.7$, and a salt/heat diffusivity ratio of $\tau = 0.01$. Asymptotic analysis was used to describe the boundary in four regions shown in figure 1, numbered 1 to 4. The asymptotics of region 4 are those found by Thorpe *et al.* (1969), while the others

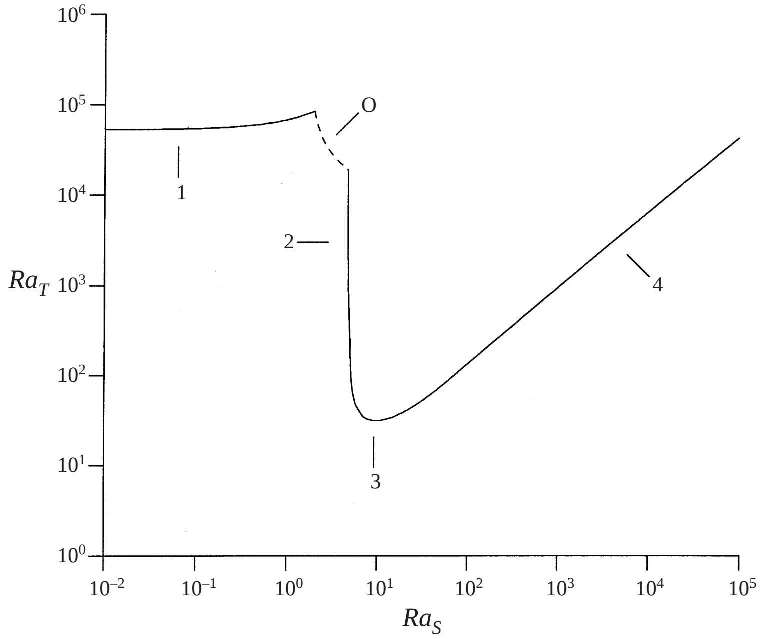

FIGURE 1. Marginal stability curve for a salinity gradient in a vertical slot with $\sigma = 6.7$ and $\tau = 0.01$. The onset of instability is steady everywhere except for the short dashed curve. The numbers indicate the sections of the curve where different asymptotic regimes hold (from Kerr & Tang 1999).

were found by KT. KT failed to find an asymptotic description for the oscillatory section of the boundary, labelled O. The origin of these oscillatory solutions was not clear.

In essence, this paper sets out to complete the work of KT by examining in more detail the oscillatory portion of the boundary. In § 2 we set out the problem. In § 3 we will describe the oscillatory branch of solutions and show how it is linked to the oscillatory or wave instabilities found in high Prandtl number fluids. The asymptotics of this section of the stability boundary are given in § 4.

2. Statement of problem

The non-dimensional linear equations for the perturbation streamfunction, ψ, temperature, T, and salinity, S, from Thangam *et al.* (1981) are

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \alpha^2\right)^2 \psi - \frac{\mathrm{i}\alpha}{\sigma}\left(\bar{W}(x)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \alpha^2\right)\psi - \psi\bar{W}''(x)\right) + Ra_T T' - Ra_S S' - \frac{\lambda}{\sigma}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \alpha^2\right)\psi = 0, \qquad (2.1a)$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \alpha^2\right) T + \mathrm{i}\alpha\psi\bar{T}_x(x) - \mathrm{i}\alpha\bar{W}(x)T - \lambda T = 0, \qquad (2.1b)$$

$$\tau\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \alpha^2\right) S + \mathrm{i}\alpha\psi\bar{S}_x(x) - \mathrm{i}\alpha\bar{W}(x)S + \psi' - \lambda S = 0, \qquad (2.1c)$$

where λ is the growth rate of the instabilities and α the vertical wavenumber. The boundary conditions for the fluid at the vertical walls are no slip for the flow, zero temperature perturbation and no salt flux, giving

$$\psi = \psi' = T = S' = 0 \quad \text{on } x = \pm 1/2. \tag{2.2}$$

The above equations have been made non-dimensional using the scalings D for length, D^2/κ_T for time, ΔT for temperature and $D|\bar{S}_z|$ for salinity, where D is the width of the slot, κ_T the thermal diffusivity, ΔT the temperature difference between the walls and $\bar{S}_z$ the constant vertical salinity gradient. The non-dimensional parameters here are the temperature and salt Rayleigh numbers

$$Ra_T = \frac{g\alpha\Delta T D^3}{\nu\kappa_T}, \quad Ra_S = \frac{g(-\beta\bar{S}_z)D^4}{\nu\kappa_T}, \tag{2.3}$$

and the Prandtl number and salt/heat diffusivity ratio

$$\sigma = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_S}{\kappa_T}. \tag{2.4}$$

Here g is the acceleration due to gravity, ν the kinematic viscosity, κ_S the salt diffusivity, α the coefficient of thermal expansion and β the coefficient of density increase with respect to the addition of salt.

The steady non-dimensional background state is given by

$$\bar{W}(x) = \frac{Ra_T}{2M^3(\sin M + \sinh M)}(\sinh M_1 \sin M_2 - \sin M_1 \sinh M_2), \tag{2.5a}$$

$$\bar{T}(x) = -x, \tag{2.5b}$$

$$\bar{S}_x(x) = -\frac{Ra_T}{4\tau M^4}\left(1 + \frac{1}{\sin M + \sinh M}(\cosh M_1 \sin M_2 - \cosh M_2 \sin M_1 - \sinh M_1 \cos M_2 + \sinh M_2 \cos M_1)\right), \tag{2.5c}$$

where

$$M = \left(\frac{Ra_S}{4\tau}\right)^{1/4}, \quad M_1 = \left(Mx + \frac{M}{2}\right), \quad M_2 = \left(Mx - \frac{M}{2}\right). \tag{2.6}$$

Here $\bar{W}(x)$, $\bar{T}(x)$ and $\bar{S}_x(x)$ are the non-dimensional vertical velocity, temperature and horizontal salinity gradient, which satisfy the boundary conditions

$$\bar{W}(\pm 1/2) = 0, \quad \bar{T}(\pm 1/2) = \mp 1/2, \quad \bar{S}_x(\pm 1/2) = 0. \tag{2.7}$$

The temperature gradient gives rise to the vertical velocity. This in turn induces a horizontal salinity gradient which will moderate the vertical velocity.

3. The oscillatory instabilities

The instabilities in the classic problem of a laterally heated vertical slot with no stratification and no salinity gradient ($Ra_S = 0$) can take the form of either steady or oscillatory convection. Korpela, Gözüm & Baxi (1973) found that for $\sigma > 12.7$ instabilities set in as travelling waves, but for lower values of the Prandtl number the initial instabilities were steady. The graph of the thermal Rayleigh number for

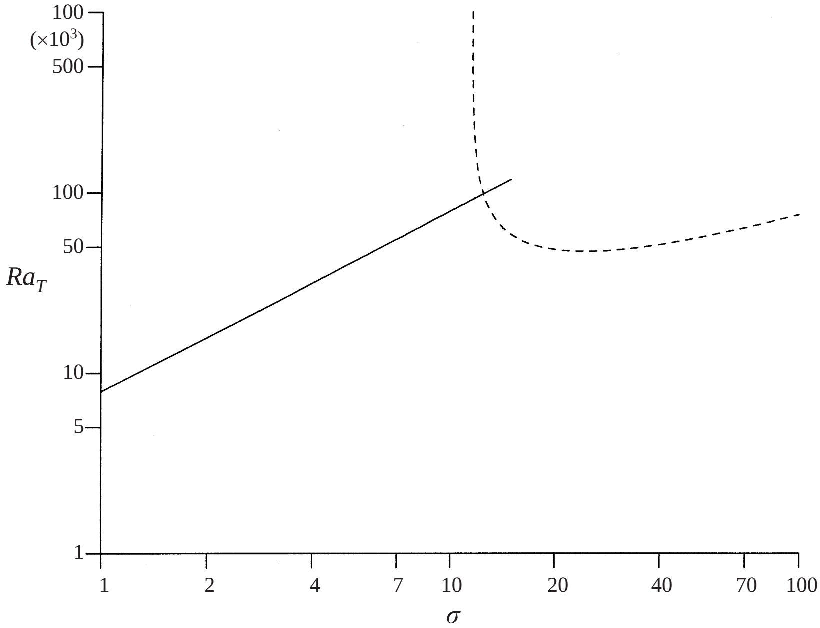

FIGURE 2. The value of Ra_T for the onset of instability in a laterally heated vertical slot with no vertical temperature or salinity gradient as a function of the Prandtl number, σ. Where the instability is steady the line is solid, and where it is oscillatory or wave-like it is dashed.

marginal stability is shown in figure 2. It is more common to use the Grashof number ($Gr = g\alpha\Delta T D^3/\nu^2$) which is a more appropriate non-dimensional parameter for steady convection in this problem. We use the Rayleigh number here for compatibility with other studies of double-diffusive convection in a vertical slot. The cross-over point between the two different modes of instability is found to be at $\sigma = 12.454$, slightly lower than the value reported by Korpela *et al.*† For Prandtl numbers less than this value the critical Rayleigh numbers lie on an almost straight line with slope 1 on this logarithmic plot, indicating that the Rayleigh number is approximately proportional to the Prandtl number (or equivalently, the Grashof number is almost constant). The critical Rayleigh number on the overstable section shows an initial steep decline reaching a minimum near $\sigma = 24.412$ before increasing towards a line of constant slope. A feature of this curve that will have some bearing on the following discussion is that the oscillatory branch of solutions tends to infinity as the Prandtl number decreases towards 11.562. For a Prandtl number less than this value there are no linear oscillatory instabilities.

We now look at how variations of the Prandtl number affect stability of a fluid in a laterally heated slot with a vertical salinity gradient. We fix the value of the salt/heat diffusivity ratio, τ, to be 0.01, a value appropriate for salt in water. The stability curves for the laterally heated slot with a vertical salinity gradient for four different Prandtl numbers are shown in figure 3: (*a*) 3.7099, (*b*) 6.7, (*c*) 12.454 and (*d*) 15.0. In each case the results are displayed for Rayleigh numbers in the ranges $0 \leqslant Ra_S \leqslant 6$ and $1000 \leqslant Ra_T \leqslant 16000$. In each graph there are three curves: a

† Bergholz (1978) also calculated the critical Rayleigh/Grashof numbers for instabilities in a vertical slot for $\sigma = 12.7$. His figure 3 clearly shows the critical Grashof numbers for the two modes differing by nearly 1000. Our calculations indicate the critical Grashof numbers for the stationary and oscillatory modes are 7873 and 6947 respectively for $\sigma = 12.7$.

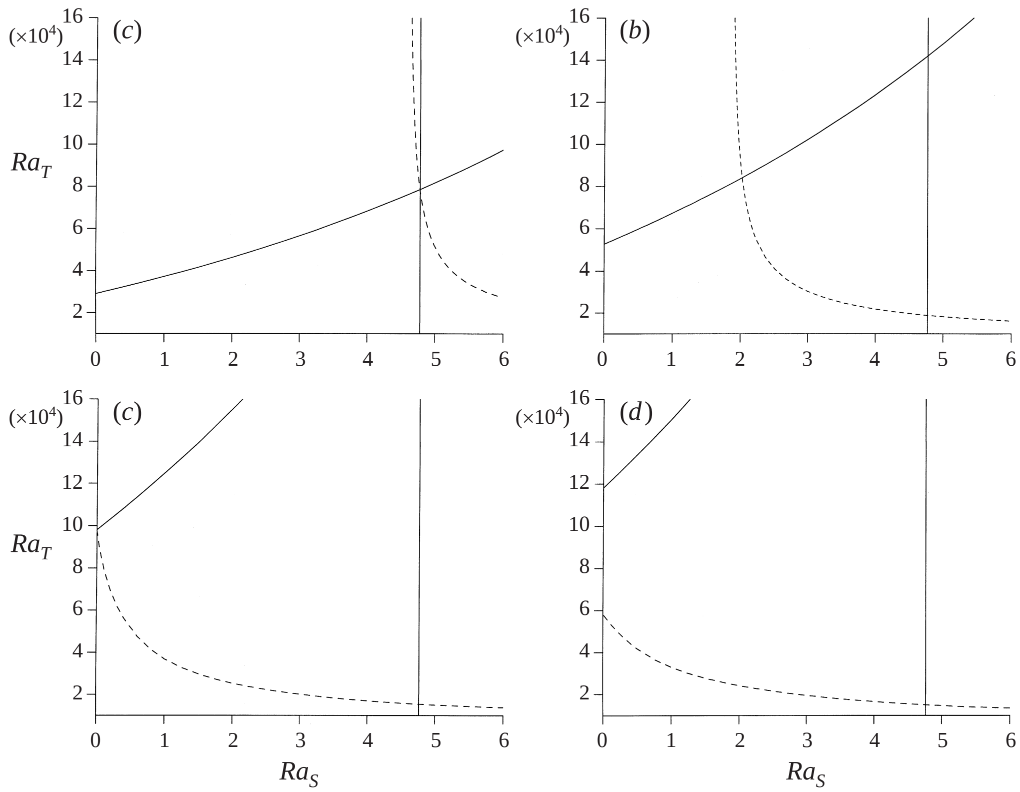

FIGURE 3. Marginal stability curves for the cases (*a*) $\sigma = 3.7099$, (*b*) $\sigma = 6.7$, (*c*) $\sigma = 12.454$ and (*d*) $\sigma = 15.0$. The dashed lines indicate where the initial instability is oscillatory and the solid lines where it is steady.

dashed curve indicating the boundary for the onset of oscillatory or travelling wave instabilities, a nearly vertical line indicating a steady instability (the large-Ra_T regime of KT), and a sloping solid line indicating another branch of steady instabilities (the small-Ra_S regime of KT). In all cases the fluid is stable in the bottom left corner. For the above Prandtl numbers the sloping steady branch of solutions has a positive slope, indicating the salinity gradient is stabilizing. The curve's point of intersection with the vertical axis steadily increases as the Prandtl number increases as is expected from the purely thermal problem. The other branch of steady solutions, the large-Ra_T branch, remains almost vertical over the range of temperature Rayleigh numbers. Its position is a weak function of the Prandtl number, showing little variation here.

The oscillatory branch of solutions only intersects the vertical axis in figures 3(*c*) and 3(*d*). Both show that the effect of a small salinity gradient is to further destabilize the fluid between the walls to oscillatory instabilities. As the Prandtl number increases from 12.454 to 15 the value of the intersection of this curve with the Ra_T-axis decreases as predicted by the purely thermal problem. As the Prandtl number decreases towards 11.562 the point of intersection with the vertical axis moves off to infinity, but the curve for larger values of the salt Rayleigh number moves up by a relatively small amount. As the Prandtl number decreases beyond this point the location of the vertical asymptote of the oscillatory branch moves to the right. Meanwhile the rest of the curve continues to move upwards.

It is clear that there are three possible regimes for the onset of instability. For low values of the Prandtl number the oscillatory mode of instabilities is not important, with only the two steady branches giving the stability boundary for the system. For $\sigma = 3.7099$ the oscillatory branch passes through the intersection of the two steady branches and so at this point there are three different modes of marginal stability with non-zero wavenumber. For values of the Prandtl number between this value and $\sigma = 12.454$ there is the second regime, which includes the case of water. In this regime the stability boundary has an initial steady section, followed by the oscillatory branch and then a second transition to the large-Ra_T regime of KT. The last regime is for Prandtl numbers greater than 12.454. The initial instabilities for low Ra_S are always oscillatory with only the one transition to the nearly vertical branch of steady instabilities.

4. Asymptotics

In order to obtain an asymptotic description of the oscillatory branch of solutions that describes the curved region in figures 1 and 3(*b*) we assume the Prandtl number is less than 11.562 and follow the oscillatory branch of solutions upwards, looking at the limit of large Ra_T. This leads to an expansion equivalent to the large-Ra_T regime of KT, but now we must include the non-zero frequency of the instabilities. As in KT we pose a large-Ra_T expansion

$$\psi(x) = \psi_0(x) + Ra_T^{-1}\psi_1(x) + Ra_T^{-2}\psi_2(x) + \cdots, \tag{4.1a}$$

$$T(x) = Ra_T^{-1}T_0(x) + Ra_T^{-2}T_1(x) + Ra_T^{-3}T_2(x) + \cdots, \tag{4.1b}$$

$$S(x) = S_0(x) + Ra_T^{-1}S_1(x) + Ra_T^{-2}S_2(x) + \cdots, \tag{4.1c}$$

$$Ra_S = Ra_{S0} + Ra_T^{-1}Ra_{S1} + Ra_T^{-2}Ra_{S2} + \cdots, \tag{4.1d}$$

$$\lambda = \mathrm{i}\left(\omega_0 + Ra_T^{-1}\omega_1 + Ra_T^{-2}\omega_2 + \cdots\right). \tag{4.1e}$$

We again note that $\bar{W}(x)$ and $\bar{S}_x(x)$ are both proportional to Ra_T at leading order. However when expanding to higher orders we must allow for the Ra_T dependence via the variation of Ra_S. Thus we will expand them as

$$\bar{W}(x) = Ra_T\bar{W}_0(x) + \bar{W}_1(x) + Ra_T^{-1}\bar{W}_2(x) + \cdots, \tag{4.2a}$$

$$\bar{S}_x(x) = Ra_T\bar{S}_{x0}(x) + \bar{S}_{x1}(x) + Ra_T^{-1}\bar{S}_{x2}(x) + \cdots. \tag{4.2b}$$

Rescaling the vertical wavenumber, $\alpha = Ra_T^{-1}\alpha_0$, gives the following leading-order set of equations:

$$\psi_0'''' - \frac{\mathrm{i}\alpha_0}{\sigma}(\bar{W}_0(x)\psi_0'' - \psi_0\bar{W}_0''(x)) + T_0' - Ra_{S0}S_0' - \frac{\mathrm{i}\omega_0\psi_0''}{\sigma} = 0, \tag{4.3a}$$

$$T_0'' - \mathrm{i}\alpha_0\psi_0 - \mathrm{i}\alpha_0\bar{W}_0(x)T_0 - \mathrm{i}\omega_0 T_0 = 0, \tag{4.3b}$$

$$\tau S_0'' + \mathrm{i}\alpha_0\psi_0\bar{S}_{x0}(x) - \mathrm{i}\alpha_0\bar{W}_0(x)S_0 + \psi_0' - \mathrm{i}\omega_0 S_0 = 0, \tag{4.3c}$$

with boundary conditions

$$\psi_0 = \psi_0' = T_0 = S_0' = 0 \quad \text{on } x = \pm 1/2. \tag{4.4}$$

These equations were solved using a modified version of the Galerkin program used to find the oscillatory solutions in KT. The minimum value of the leading-order salt

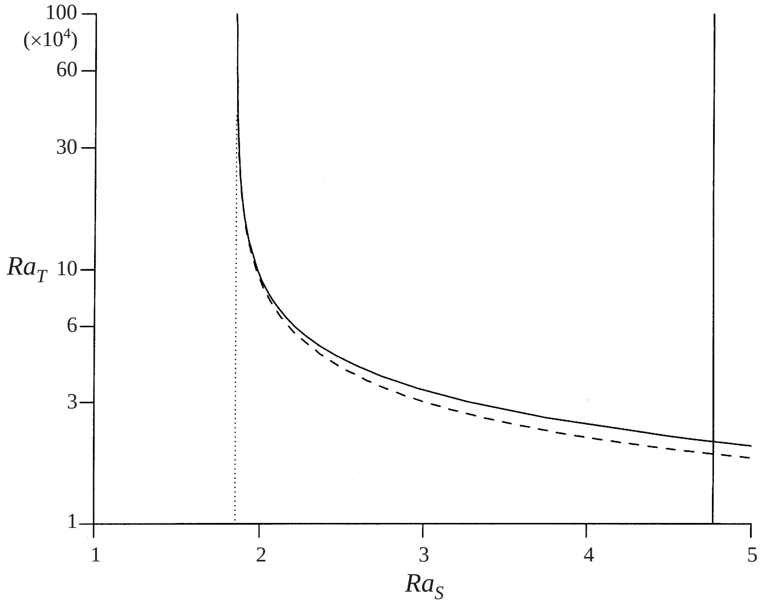

FIGURE 4. A comparison of the marginal stability curve of KT for the full problem (dashed curve) and the asymptotic approximation for the oscillatory instabilities for $\sigma = 6.7$ and $\tau = 0.01$. The leading-order asymptotic approximation is indicated by the vertical dotted line and the large-Ra_T steady branch by the near vertical solid line.

Rayleigh number and the corresponding wavenumber and frequency is found to be

$$Ra_{S0} = 1.8548, \quad \alpha_0 = 48780, \quad \omega_0 = 257.75. \tag{4.5}$$

This result gives a vertical boundary for the oscillatory solutions. However, this is not observed in the (Ra_T, Ra_S)-plane, where the boundary is always curved. To get a better description of this boundary we have to look to higher orders in our expansion.

If we look at the $O(Ra_T^{-1})$ equations and apply a solvability condition we find that $Ra_{S1} = \omega_1 = 0$. We omit the details and continue directly with the $O(Ra_T^{-2})$ equations:

$$\psi_2'''' - \frac{i\alpha_0}{\sigma}(\bar{W}_0(x)\psi_2'' - \psi_2\bar{W}_0''(x)) + T_2' - Ra_{S0}S_2' - \frac{i\omega_0\psi_2''}{\sigma} = 2\alpha_0^2\psi_0'' - \frac{i\alpha_0^3}{\sigma}\bar{W}_0(x)\psi_0 + \frac{i\alpha_0}{\sigma}(\bar{W}_2(x)\psi_0'' - \psi_0\bar{W}_2''(x)) + i\omega_0\alpha_0^2\psi_0 + i\omega_2\psi_0'' + Ra_{S2}S_0', \tag{4.6a}$$

$$T_2'' - i\alpha_0\psi_2 - i\alpha_0\bar{W}_0(x)T_2 - i\omega_0 T_2 = \alpha_0^2 T_0 + i\alpha_0\bar{W}_2(x)T_0 + i\omega_2 T_0, \tag{4.6b}$$

$$\tau S_2'' + i\alpha_0\psi_2\bar{S}_{x0}(x) - i\alpha_0\bar{W}_0(x)S_2 + \psi_2' - i\omega_0 S_2 = \tau\alpha_0^2 S_0 - i\alpha_0\psi_0\bar{S}_{x2}(x) + i\omega_2 S_0. \tag{4.6c}$$

Applying a solvability condition to these yields a complex linear equation with two real unknowns, Ra_{S2} and ω_2. For $\sigma = 6.7$ and $\tau = 0.01$ this has the solution

$$Ra_{S2} = 1.283 \times 10^9, \quad \omega_2 = -4.6905 \times 10^{10}. \tag{4.7}$$

The asymptotic result for the salt Rayleigh number is compared with the oscillatory branch of instability for the full problem in figure 4. Also plotted in this figure is the asymptotic prediction for the steady large-Ra_T branch from KT (the second-order correction $Ra_{S2} = 47842$ has no visible effect on the curve with the range and scale of this figure). It can be seen that the asymptotics match the solution to the full problem very well for large values of Ra_T, and that the agreement between the two is quite close all the way to where the two curves cross the steady large-Ra_T branch. Thus the asymptotics to this order provide a reasonable description of the curved part of the boundary for all cases where it represents the primary mode of marginal instability.

5. Conclusions

In conclusion, the oscillatory branch of instabilities observed by Young & Rosner and by KT has its origins in the oscillatory instabilities in a vertical unstratified laterally heated vertical slot. The presence of a vertical salinity gradient tends to destabilize this mode of instability. The salinity gradient's effect is sufficiently strong that it can sometimes bring this mode of instability back into existence for lower values of the Prandtl number when the mode no longer exists in an unstratified slot. This is the case for Prandtl numbers appropriate for water as used by Young & Rosner and KT.

The asymptotics for the oscillatory branch now completes the set of asymptotics of the five different primary modes of instability, giving a good description of nearly all the linear stability boundary for the problem of heating a salt-stratified fluid in a vertical slot. This oscillatory regime is derived using the same set of equations as the large-Ra_T regime of KT. The steady instabilities extended across the whole slot, and were driven by the lateral salinity gradients induced by the temperature difference. The presence of the shear was stabilizing. The oscillatory instabilities are similar, except that they are mainly located in either the upward or downward flowing regions of the fluid, which gives rise to their wave-like behaviour.

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